

Another proof of the local curvature estimate for the Ricci flow

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Abstract

By using the De Giorgi iteration method we will give a new simple proof of the recent result of B. Kotschwar, O. Munteanu, J. Wang [KMW] and N. Sesum [S] on the local boundedness of the Riemannian curvature tensor of solutions of Ricci flow in terms of its initial value on a given ball and a local uniform bound on the Ricci curvature.

Key words: Ricci flow, local boundedness, Riemannian curvature, Ricci curvature

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1 Introduction

There is a lot of interest on Ricci flow ([CK], [CLN], [H2], [MF], [MT]) because it is a very powerful tool in the study of the geometry of manifolds. Recently G. Perelman [P1], [P2], by using the Ricci flow technique solved the famous Poincare conjecture in geometry. Let $(M, g(t))$, $0 < t < T$, be a n -dimensional Riemannian manifold. We say that the metric $g(t) = g_{ij}(t)$ evolves by the Ricci flow if it satisfies

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \quad (1.1)$$

in $(0, T)$ where R_{ij} is the Ricci curvature of the metric $g(t)$. Short time existence of solution of Ricci flow on compact Riemannian manifolds with any initial metric at $t = 0$ was

proved by R. Hamilton in [H1]. Short time existence of solution of Ricci flow on complete non-compact manifolds with bounded curvature initial metric at time $t = 0$ was proved by W.X. Shi in [Sh1], [Sh2]. When M is a compact manifold, R. Hamilton [H1] proved that either the Ricci flow solution exists globally or there exists a maximal existence time $0 < T < \infty$ for the solution of Ricci flow and

$$\lim_{t \nearrow T} |Rm|_{g(t)} = \infty.$$

Hence in order to know whether the solution of Ricci flow can be extended beyond its interval of existence $(0, T)$, it is important to prove boundedness of the Riemannian curvature for the solution of Ricci flow near the time T . Uniform boundedness of the Riemannian curvature of the solution of Ricci flow on a compact manifold when the solution has uniform bounded Ricci curvature on $(0, T)$ was proved by N. Sesum in [S] using a blow-up contradiction argument and G. Perelman's noncollapsing result [P1]. Local boundedness of the Riemannian curvature for κ -noncollapsing solutions of Ricci flow in term of its local $L^{\frac{n}{2}}$ norm when its local $L^{\frac{n}{2}}$ norm is sufficiently small was also proved by R. Ye in [Y1], [Y2], using Moser iteration technique and the point picking technique of G. Perelman [P1]. Similar result was also obtained by X Dai, G Wei and R Ye in [DWY].

Local boundedness of the Riemannian curvature of the solution of Ricci flow in terms of its initial value on a given ball and a local uniform bound on the Ricci curvature was proved by B. Kotschwar, O. Munteanu, J. Wang using Moser iteration technique and results of P. Li [L] in [KMW]. A similar local Riemannian curvature result was proved recently by C.W. Chen [C] using the point picking technique of Perelman [P1], M.T. Anderson's harmonic coordinates [A] and elliptic regularity results [GT]. In this paper we will use the De Giorgi iteration method to give a new simple proof of this result.

We will assume that $(M, g(t))$ is a smooth solution of the Ricci flow (1.1) in $[0, T)$ for the rest of the paper. For any $x_0 \in M$, $\rho > 0$ and $0 \leq t < T$, we let $B_{g(t)}(x_0, \rho) = \{x \in M : \text{dist}_{g(t)}(x, x_0) < \rho\}$, $V_{x_0}(\rho, t) = \text{vol}_{g(t)}(B_{g(t)}(x_0, \rho))$, $V_{x_0}(\rho) = V_{x_0}(\rho, 0)$, $|Ric|(x, t) = |Ric(x, t)|_{g(t)}$ and $|Rm|(x, t) = |Rm(x, t)|_{g(t)}$. We let dv_t be the volume element of the metric $g(t)$ and let $C > 0$ denote a generic constant that may change from line to line. Note that by Corollary 13.3 of [H1] or Lemma 7.4 of [CK],

$$\frac{\partial}{\partial t} |Rm|^2 \leq \Delta |Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3 \quad (1.2)$$

in $(0, T)$ for some constant $C > 0$ depending only on n . Since $|\nabla |Rm|| \leq |\nabla Rm|$, by (1.2),

$$\frac{\partial}{\partial t} |Rm| \leq \Delta |Rm| + C|Rm|^2 \quad \text{in } (0, T). \quad (1.3)$$

We will prove the following main result in this paper.

Theorem 1.1. (cf. Theorem 1 of [KMW]) *Let $g(t)$, $0 \leq t < T$, be a smooth solution of Ricci flow on a n -dimensional Riemannian manifold M . Suppose there exists $x_0 \in M$ and constants $K > 0$,*

$\rho > 0$, such that

$$|\text{Ric}| \leq K \quad \text{in } B_{g(0)}\left(x_0, \frac{2\rho}{\sqrt{K}}\right) \times [0, T] \quad (1.4)$$

and

$$\Lambda_0 := \sup_{B_{g(0)}\left(x_0, \frac{2\rho}{\sqrt{K}}\right)} |Rm|(x, 0) < \infty. \quad (1.5)$$

Then for any $n \geq 3$ and $p > \frac{n+2}{2}$ there exist constants $C_0 > 0$ and $C > 0$ such that

$$\begin{aligned} & |Rm|(x, t) \\ & \leq C_0 \left\{ \frac{\rho^{\frac{2n}{n+2}} e^{C(\rho\sqrt{K}+tK)}}{K^{\frac{n}{n+2}} V_{x_0}\left(\rho/\sqrt{K}\right)^{\frac{2}{n+2}} \min(t, \rho^2/K)} \left[(K + E_p(t))t + 1\right] \right\}^{\frac{n+2}{2p}} (1 + \sqrt{tV_{x_0}(2\rho/\sqrt{K})} E_p(t)^{\frac{1}{2}})^{\frac{n+4}{2p}} \end{aligned}$$

holds for any $x \in B_{g(0)}(x_0, \rho/\sqrt{K})$ and $0 < t < T$ where

$$E_p(t) = C e^{CKt} t \left[\Lambda_0^{2p} V_{x_0}(2\rho/\sqrt{K}) + K^{2p} (1 + \rho^{-4p}) V_{x_0}(\rho/\sqrt{K}) \right]$$

and for $n = 2$ and any $p > \frac{5}{2}$ there exist constants $C_0 > 0$ and $C > 0$ such that

$$\begin{aligned} & |Rm|(x, t) \\ & \leq C_0 \left\{ \frac{\rho^{\frac{4}{5}} e^{C(\rho\sqrt{K}+tK)}}{K^{\frac{2}{5}} V_{x_0}\left(\rho/\sqrt{2K}\right)^{\frac{2}{5}} \min(t, \rho^2/K)} \left[(K + E_p(t))t + 1\right] \right\}^{\frac{5}{2p}} (1 + (4\rho/\sqrt{K}) \sqrt{tV_{x_0}(2\rho/\sqrt{K})} E_p(t)^{\frac{1}{2}})^{\frac{7}{2p}} \end{aligned}$$

holds for any $x \in B_{g(0)}(x_0, \rho/\sqrt{K})$ and $0 < t < T$.

2 The main result

We first recall a result of [KMW]:

Proposition 2.1. (Proposition 1 of [KMW]) Let $g(t)$, $0 \leq t < T$, be a smooth solution of Ricci flow on a n -dimensional Riemannian manifold M . Suppose there exists $x_0 \in M$ and constants $K > 0$, $\rho > 0$, such that (1.4) holds. Then for any $n \geq 2$ and $p \geq 3$ there exists a constant $c = c(n, p) > 0$ such that

$$\int_{B_{g(0)}\left(x_0, \frac{\rho}{\sqrt{K}}\right)} |Rm|(x, t)^p dv_t \leq c e^{CKt} \left\{ \int_{B_{g(0)}\left(x_0, \frac{2\rho}{\sqrt{K}}\right)} |Rm|(x, 0)^p dv_0 + c K^p (1 + \rho^{-2p}) V_{x_0}\left(\rho/\sqrt{K}, t\right) \right\}$$

holds for any $0 \leq t < T$.

Theorem 2.2. Let $g(t)$, $0 \leq t < T$, be a smooth solution of Ricci flow on a n -dimensional Riemannian manifold M . Suppose there exists $x_0 \in M$ and constants $K > 0$, $\rho > 0$, such that (1.4) holds. Then for any $n \geq 3$ and $p > \frac{n+2}{2}$ there exist constants $C_0 > 0$ and $C > 0$ such that

$$|Rm|(x, t) \leq C_0 \left\{ \frac{\rho^{\frac{2n}{n+2}} e^{C(\rho \sqrt{K} + tK)}}{K^{\frac{n}{n+2}} V_{x_0}(\rho / \sqrt{K})^{\frac{2}{n+2}} \min(t, \rho^2 / K)} \left[\left(\iint_{Q_0} |Rm|^{2p} dv_0 dt + K \right) t + 1 \right] \right\}^{\frac{n+2}{2p}} \cdot \left(1 + \iint_{Q_0} |Rm|^p dv_0 dt \right)^{\frac{n+4}{2p}} \quad (2.1)$$

holds for any $x \in B_{g(0)}(x_0, \rho / \sqrt{K})$ and $0 < t < T$ where $Q_0 = B_{g(0)}(x_0, 2\rho / \sqrt{K}) \times (t/4, t)$ and for $n = 2$ and any $p > \frac{5}{2}$ there exist constants $C_0 > 0$ and $C > 0$ such that

$$|Rm|(x, t) \leq C_0 \left\{ \frac{\rho^{\frac{4}{5}} e^{C(\rho \sqrt{K} + tK)}}{K^{\frac{2}{5}} V_{x_0}(\rho / \sqrt{2K})^{\frac{2}{5}} \min(t, \rho^2 / K)} \left[\left(K + \frac{4\rho}{\sqrt{K}} \iint_{Q_0} |Rm|^{2p} dv_0 dt \right) t + 1 \right] \right\}^{\frac{5}{2p}} \cdot \left(1 + \frac{4\rho}{\sqrt{K}} \iint_{Q_0} |Rm|^p dv_0 dt \right)^{\frac{7}{2p}} \quad (2.2)$$

holds for any $x \in B_{g(0)}(x_0, \rho / \sqrt{K})$ and $0 < t < T$.

Proof. **Case 1:** $n \geq 3$.

Let $v = |Rm|$, $0 < t < T$ and $p > \frac{n+2}{2}$. We will use a modification of the proof of Proposition 2.1 of [DDD] to prove this theorem. By (1.4),

$$\begin{aligned} e^{-2Kt} g_{ij}(x, 0) &\leq g(x, s) \leq e^{2Kt} g_{ij}(x, 0) \quad \forall x \in B_{g(0)}\left(x_0, \frac{2\rho}{\sqrt{K}}\right), 0 \leq s \leq t < T \\ \Rightarrow e^{-nKt} dv_0 &\leq dv_s \leq e^{nKt} dv_0 \quad \forall x \in B_{g(0)}\left(x_0, \frac{2\rho}{\sqrt{K}}\right), 0 \leq s \leq t < T. \end{aligned} \quad (2.3)$$

Let $\rho_m = (\rho / \sqrt{K})(1 + 2^{-m})$ and $t_m = (1 - 2^{-m-1})t/2$ for any $m \in \mathbb{Z}^+$. Then $\rho_0 = 2\rho / \sqrt{K}$ and $t_0 = t/4$. Moreover ρ_m decreases to ρ / \sqrt{K} and t_m increases to $t/2$ as $m \rightarrow \infty$. Let $B_{\rho_m} = B_{g(0)}(x_0, \rho_m)$, $Q_m = B_{\rho_m} \times (t_m, t)$ and $Q_m^s = B_{\rho_m} \times (t_m, s)$ for any $t_m \leq s \leq t$. Then

$$B_{2\rho/\sqrt{K}} \times (t/4, t) = Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_{m-1} \supseteq Q_m \supseteq \cdots \supseteq Q_\infty = B_{\rho/\sqrt{K}} \times (t/2, t).$$

We choose a sequence of Lipschitz continuous functions $\{\phi_m\}$ on $M \times (0, t)$ such that $0 \leq \phi_m \leq 1$ on $M \times (0, t)$, $\phi_m(x, s) = 1$ for $(x, s) \in Q_{m+1}$, $\phi_m(x, s) = 0$ for $(x, s) \in M \times (0, t) \setminus Q_m$, and satisfying

$$\begin{cases} |\nabla \phi_m| \leq \frac{C \sqrt{K} 2^m}{\rho} & \text{in } Q_m \\ 0 \leq \phi_{m,t} \leq \frac{C 2^m}{t} & \text{in } Q_m. \end{cases} \quad (2.4)$$

Let $k > 0$ be a constant to be determined later and $k_m = k(1 - 2^{-m})$ for any $m \in \mathbb{Z}^+$. Multiplying (1.3) by $(v - k_{m+1})_+^{p-1} \phi_m^2$ and integrating over Q_m^s , $t_m \leq s \leq t$,

$$\begin{aligned}
& \frac{1}{p} \iint_{Q_m^s} \phi_m^2 \frac{\partial}{\partial t} (v - k_{m+1})_+^p dv_t dt \\
& + \iint_{Q_m^s} \nabla(v - k_{m+1})_+ \cdot [(p-1)(v - k_{m+1})_+^{p-2} \phi_m^2 \nabla(v - k_{m+1})_+ + 2(v - k_{m+1})_+^{p-1} \phi_m \nabla \phi_m] dv_t dt \\
& \leq C \iint_{Q_m^s} v^2 (v - k_{m+1})_+^{p-1} \phi_m^2 dv_t dt \\
& \leq C \left(\iint_{Q_0} v^{2p} dv_t dt \right)^{\frac{1}{p}} \left(\iint_{Q_m^s} (v - k_{m+1})_+^p \phi_m^2 dv_t dt \right)^{\frac{p-1}{p}}. \tag{2.5}
\end{aligned}$$

Since $\frac{d}{dt}(dv_t) = -R dv_t$, by (1.4),

$$\begin{aligned}
& \iint_{Q_m^s} \phi_m^2 \frac{\partial}{\partial t} (v - k_{m+1})_+^p dx dt \\
& = \int_{t_m}^s \frac{d}{dt} \left(\int_{B_{\rho_m}} (v - k_{m+1})_+^p \phi_m^2 dv_t \right) dt - 2 \iint_{Q_m^s} (v - k_{m+1})_+^p \phi_m \phi_{m,t} dv_t dt \\
& \quad + \iint_{Q_m^s} (v - k_{m+1})_+^p \phi_m^2 R dv_t dt \\
& \geq \int_{B_{\rho_m}} (v(x, s) - k_{m+1})_+^p \phi_m(x, s)^2 dv_s - C \frac{2^m}{t} \iint_{Q_m^s} (v - k_{m+1})_+^p dv_t dt \\
& \quad - CK \iint_{Q_m^s} (v - k_{m+1})_+^p \phi_m^2 dv_t dt. \tag{2.6}
\end{aligned}$$

Since

$$\int_{B_{\rho_m}} |\nabla(\phi_m (v - k_{m+1})_+^{\frac{p}{2}})|^2 dv_t \leq \frac{11}{10} \int_{B_{\rho_m}} \phi_m^2 |\nabla(v - k_{m+1})_+^{\frac{p}{2}}|^2 dv_t + 11 \int_{B_{\rho_m}} (v - k_{m+1})_+^p |\nabla \phi_m|^2 dv_t$$

and

$$\begin{aligned}
& 2 \int_{B_{\rho_m}} \phi_m (v - k_{m+1})_+^{p-1} \nabla \phi_m \cdot \nabla(v - k_{m+1})_+ dv_t \\
& = \frac{4}{p} \int_{B_{\rho_m}} \phi_m (v - k_{m+1})_+^{\frac{p}{2}} \nabla \phi_m \cdot \nabla(v - k_{m+1})_+^{\frac{p}{2}} dv_t \\
& \geq -\frac{2}{p} \left(\frac{p-1}{p} \int_{B_{\rho_m}} \phi_m^2 |\nabla(v - k_{m+1})_+^{\frac{p}{2}}|^2 dv_t + \frac{p}{p-1} \int_{B_{\rho_m}} (v - k_{m+1})_+^p |\nabla \phi_m|^2 dv_t \right),
\end{aligned}$$

$$\begin{aligned}
& (p-1) \int_{B_{\rho_m}} \phi_m^2 (v - k_{m+1})_+^{p-2} |\nabla (v - k_{m+1})_+|^2 dv_t + 2 \int_{B_{\rho_m}} \phi_m (v - k_{m+1})_+^{p-1} \nabla \phi_m \cdot \nabla (v - k_{m+1})_+ dv_t \\
& \geq \frac{4(p-1)}{p^2} \int_{B_{\rho_m}} \phi_m^2 |\nabla (v - k_{m+1})_+^{\frac{p}{2}}|^2 dv_t \\
& \quad - \frac{2}{p} \left(\frac{p-1}{p} \int_{B_{\rho_m}} \phi_m^2 |\nabla (v - k_{m+1})_+^{\frac{p}{2}}|^2 dv_t + \frac{p}{p-1} \int_{B_{\rho_m}} (v - k_{m+1})_+^p |\nabla \phi_m|^2 dv_t \right) \\
& \geq \frac{2(p-1)}{p^2} \left(\frac{10}{11} \int_{B_{\rho_m}} |\nabla (\phi_m (v - k_{m+1})_+^{\frac{p}{2}})|^2 dv_t - 10 \int_{B_{\rho_m}} (v - k_{m+1})_+^p |\nabla \phi_m|^2 dv_t \right) \\
& \quad - \frac{2}{p-1} \int_{B_{\rho_m}} (v - k_{m+1})_+^p |\nabla \phi_m|^2 dv_t \\
& = \frac{20(p-1)}{11p^2} \int_{B_{\rho_m}} |\nabla (\phi_m (v - k_{m+1})_+^{\frac{p}{2}})|^2 dv_t - \frac{CK4^m}{\rho^2} \left(\frac{20(p-1)}{p^2} + \frac{2}{p-1} \right) \int_{B_{\rho_m}} (v - k_{m+1})_+^p dv_t.
\end{aligned} \tag{2.7}$$

By (2.5), (2.6) and (2.7),

$$\begin{aligned}
& \int_{B_{\rho_m}} v(x, s)^p \phi_m(x, s)^2 dv_s + \iint_{Q_m^s} |\nabla (\phi_m (v - k_{m+1})_+^{\frac{p}{2}})|^2 dv_t dt \\
& \leq C \left(\iint_{Q_0} v^{2p} dv_t dt \right)^{\frac{1}{p}} \left(\iint_{Q_m^s} (v - k_{m+1})_+^p dv_t dt \right)^{\frac{p-1}{p}} + C \left(K + \frac{2^m}{t} + \frac{K4^m}{\rho^2} \right) \iint_{Q_m^s} (v - k_{m+1})_+^p dv_t dt
\end{aligned} \tag{2.8}$$

By (2.3) and (2.8),

$$\begin{aligned}
& \sup_{t_m \leq s \leq t} \int_{B_{\rho_m}} v(x, s)^p \phi_m(x, s)^2 dv_0 + \iint_{Q_m} |\nabla (\phi_m (v - k_{m+1})_+^{\frac{p}{2}})|_{g(0)}^2 dv_0 dt \\
& \leq C e^{CKt} \left\{ A_1 Y_m^{\frac{p-1}{p}} + \left(K + \frac{2^m}{t} + \frac{K4^m}{\rho^2} \right) Y_m \right\}
\end{aligned} \tag{2.9}$$

where

$$A_1 = \left(\iint_{Q_0} v^{2p} dv_0 dt \right)^{\frac{1}{p}}$$

and

$$Y_m = \iint_{Q_m} (v - k_m)_+^p dv_0 dt.$$

By Theorem 14.3 of [L],

$$\begin{aligned} \int_{B_{\rho_m}} |\nabla(\phi_m(v - k_m)_+^{\frac{p}{2}})|_{g(0)}^2 dv_0 &\geq \frac{CV_{x_0}(\rho_m)^{\frac{2}{n}}}{\rho_m^2 e^{C\rho_m \sqrt{K}}} \left(\int_{B_{\rho_m}} [\phi_m(v - k_m)_+^{\frac{p}{2}}]^{\frac{2n}{n-2}} dv_0 \right)^{\frac{n-2}{n}} \\ &\geq \frac{CKV_{x_0}(\rho/\sqrt{K})^{\frac{2}{n}}}{\rho^2 e^{C\rho \sqrt{K}}} \left(\int_{B_{\rho_m}} [\phi_m(v - k_m)_+^{\frac{p}{2}}]^{\frac{2n}{n-2}} dv_0 \right)^{\frac{n-2}{n}}. \end{aligned} \quad (2.10)$$

By the Holder inequality, (2.10) and an argument similar to the proof of proposition 3.1 of chapter 1 of [D],

$$\begin{aligned} &\iint_{Q_m} [\phi_m(v - k_m)_+^{\frac{p}{2}}]^{2+\frac{4}{n}} dv_0 dt \\ &\leq \frac{C\rho^2 e^{C\rho \sqrt{K}}}{KV_{x_0}(\rho/\sqrt{K})^{\frac{2}{n}}} \left(\iint_{Q_m} |\nabla(\phi_m(v - k_m)_+^{\frac{p}{2}})|_{g(0)}^2 dv_0 dt \right) \cdot \left(\sup_{t_n \leq s \leq t} \int_{B_{\rho_m}} (v - k_m)_+^p \phi_m^2 dv_0 \right)^{\frac{2}{n}}. \end{aligned} \quad (2.11)$$

By the Holder inequality,

$$\begin{aligned} Y_{m+1} &= \iint_{Q_{m+1}} (v - k_{m+1})_+^p dv_0 dt \\ &\leq \iint_{Q_m} (v - k_{m+1})_+^p \phi_m^2 dv_0 dt \\ &\leq \left(\iint_{Q_m} [\phi_m^2 (v - k_m)_+^p]^{\frac{n+2}{n}} dv_0 dt \right)^{\frac{n}{n+2}} |E_m|^{\frac{2}{n+2}} \end{aligned} \quad (2.12)$$

where $E_m = \{(x, s) \in Q_m : v(x, s) > k_{m+1}\}$. By (2.9), (2.11) and (2.12),

$$Y_{m+1} \leq \frac{C\rho^{\frac{2n}{n+2}} e^{C(\rho \sqrt{K} + tK)}}{K^{\frac{n}{n+2}} V_{x_0}(\rho/\sqrt{K})^{\frac{2}{n+2}}} \left\{ A_1 Y_m^{\frac{p-1}{p}} + \left(K + \frac{2^m}{t} + \frac{K4^m}{\rho^2} \right) Y_m \right\} |E_m|^{\frac{2}{n+2}}. \quad (2.13)$$

Now similar to the proof on P.645 of [DDD],

$$|E_m| \leq \frac{2^{(m+1)p}}{k^p} Y_m. \quad (2.14)$$

Hence by (2.13) and (2.14),

$$\begin{aligned} Y_{m+1} &\leq \frac{C\rho^{\frac{2n}{n+2}} e^{C(\rho \sqrt{K} + tK)}}{K^{\frac{n}{n+2}} V_{x_0}(\rho/\sqrt{K})^{\frac{2}{n+2}}} \left(\frac{2^{mp}}{k^p} \right)^{\frac{2}{n+2}} \left\{ A_1 Y_m^{1+\frac{2}{n+2}-\frac{1}{p}} + \left(K + \frac{2^m}{t} + \frac{K4^m}{\rho^2} \right) Y_m^{1+\frac{2}{n+2}} \right\} \\ &\leq \frac{C_1(A_1 t + Kt + 1) \rho^{\frac{2n}{n+2}} e^{C(\rho \sqrt{K} + tK)}}{k^{\frac{2p}{n+2}} K^{\frac{n}{n+2}} V_{x_0}(\rho/\sqrt{K})^{\frac{2}{n+2}} \min(t, \rho^2/K)} \cdot b^m \max\left(Y_m^{1+\alpha}, Y_m^{1+\frac{2}{n+2}}\right) \end{aligned} \quad (2.15)$$

for some constant $C_1 > 0$ where $b = 4 \cdot 2^{\frac{2p}{n+2}}$ and $\alpha = \frac{2}{n+2} - \frac{1}{p}$. Then $0 < \alpha < \frac{2}{n+2}$. We now let $\beta > 1/\alpha$ and

$$k = \left\{ \frac{C_1(A_1 t + K t + 1) \rho^{\frac{2n}{n+2}} e^{C(\rho \sqrt{K} + tK)} b^\beta}{K^{\frac{n}{n+2}} V_{x_0}(\rho / \sqrt{K})^{\frac{2}{n+2}} \min(t, \rho^2/K)} \right\}^{\frac{n+2}{2p}} \left(1 + \int_{t/4}^t \int_{B_{2\rho/\sqrt{K}}} v^p dv_0 dt \right)^{\frac{n+4}{2p}}. \quad (2.16)$$

We claim that

$$Y_{m+1} \leq b^{-\frac{(\alpha\beta-1)[(1+\alpha)^{m+1}-(1+\alpha)]}{\alpha^2} - \frac{m}{\alpha}} < 1 \quad \forall m \in \mathbb{Z}^+. \quad (2.17)$$

In order to prove this claim we observe first that by (2.15) and (2.16),

$$Y_1 \leq b^{-\beta} < 1. \quad (2.18)$$

By (2.15), (2.16) and (2.18),

$$Y_2 \leq \frac{C_1(A_1 t + K t + 1) \rho^{\frac{2n}{n+2}} e^{C(\rho \sqrt{K} + tK)}}{k^{\frac{2p}{n+2}} K^{\frac{n}{n+2}} \min(t, \rho^2/K)} \cdot b Y_1^{1+\alpha} \leq b^{1-\beta-\beta(1+\alpha)} < 1. \quad (2.19)$$

Repeating the above argument we get,

$$Y_{m+1} \leq b^{\sum_{i=1}^m i(1+\alpha)^{m-i} - \beta \sum_{i=1}^m (1+\alpha)^i}$$

and (2.17) follows. Letting $m \rightarrow \infty$ in (2.17),

$$\lim_{m \rightarrow \infty} Y_{m+1} = 0$$

and (2.1) follows.

Case 2: $n = 2$.

Let $\bar{M} = M \times \mathbb{R}$, $\bar{g}(x, t) = g(x, t) + dx^2$, and let \bar{Rm} , \bar{Ric} , \bar{R} , be the Riemannian curvature, Ricci curvature and scalar curvature of $(\bar{M}, \bar{g}(t))$. Then

$$|\bar{Rm}|(x, y, t) = |Rm|(x, t), \quad |\bar{Ric}|(x, y, t) = |Ric|(x, t), \quad \bar{R}(x, y, t) = R(x, t) \quad (2.20)$$

for all $x \in M$, $y \in \mathbb{R}$, $0 \leq t < T$ and

$$\frac{\partial \bar{g}_{ij}}{\partial t} = -2\bar{R}_{ij} \quad \text{in } (0, T).$$

Let $\bar{V}_{x_0}(r) = \text{vol}_{\bar{g}(0)}(B_{\bar{g}(0)}((x_0, 0), r))$ for any $r > 0$ and $d\bar{v}_0 = dv_0 dy$ be the volume element of $\bar{g}(0)$. By case 1,

$$\begin{aligned} |\bar{Rm}|(x, y, t) \leq C_0 & \left\{ \frac{\rho^{\frac{6}{5}} e^{C(\rho \sqrt{K} + tK)}}{K^{\frac{3}{5}} \bar{V}_{(x_0, 0)}(\rho / \sqrt{K})^{\frac{2}{5}} \min(t, \rho^2/K)} \left[\left(\iint_{\bar{Q}_0} |Rm|^{2p} d\bar{v}_0 dt + K \right) t + 1 \right] \right\}^{\frac{5}{2p}} \\ & \cdot \left(1 + \iint_{\bar{Q}_0} |Rm|^p d\bar{v}_0 dt \right)^{\frac{7}{2p}} \end{aligned} \quad (2.21)$$

holds for any $(x, y) \in B_{\tilde{g}(0)}((x_0, 0), \rho/\sqrt{K})$ and $0 < t < T$ where $\tilde{Q}_0 = B_{\tilde{g}(0)}((x_0, 0), 2\rho/\sqrt{K}) \times (t/4, t)$. Since $B_{\tilde{g}(0)}((x_0, 0), \rho/\sqrt{K}) \supset B_{g(0)}(x_0, \rho/\sqrt{2K}) \times (-\rho/\sqrt{2K}, \rho/\sqrt{2K})$,

$$\tilde{V}_{(x_0, 0)}(\rho/\sqrt{K}) \geq (\sqrt{2}\rho/\sqrt{K}) V_{x_0}(\rho/\sqrt{2K}). \quad (2.22)$$

Since $B_{\tilde{g}(0)}((x_0, 0), 2\rho/\sqrt{K}) \subset B_{g(0)}(x_0, 2\rho/\sqrt{K}) \times (-2\rho/\sqrt{K}, 2\rho/\sqrt{K})$, by (2.20), (2.21) and (2.22), we get (2.2) and the theorem follows. \square

Remark 2.3. By Proposition 2.1, Theorem 2.2, Holder's inequality and (2.3), Theorem 1.1 follows.

Remark 2.4. By a similar argument as the proof of Theorem 2.2 for the case $n = 2$ one can extend the results of Theorem 2.2 and Theorem 1.1 to the case $n = 1$.

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